Local approximation of the Maximum Cut in regular graphs

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Local/Distributed algorithms

The distributed models on graphs:

- Synchronous rounds, at each round each vertex can send/receive messages to/from its neighbors in the graph
- No failure
- Infinite local computational power

LOCAL model:

- Unique IDs
- Messages of unlimited size

CONGEST model:

- Unique IDs
- Messages of $O(\log n)$ bits

PO model:

- Port numbering+edge orientation
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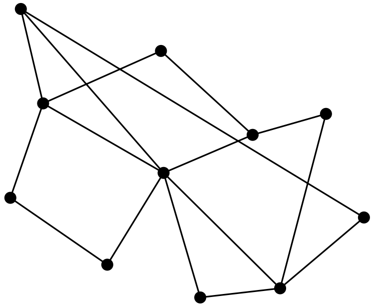
How well can we approximate the Maximum Cut in any of these models ?

The Maximum (directed) Cut in the distributed model

Problem definition: partition the vertex set into 2 parts, A and B, maximizing the number of crossing edges (crossing specifically from A to B in the directed case)



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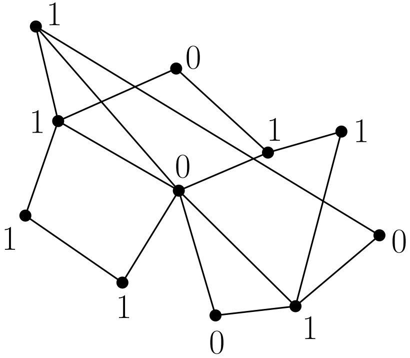

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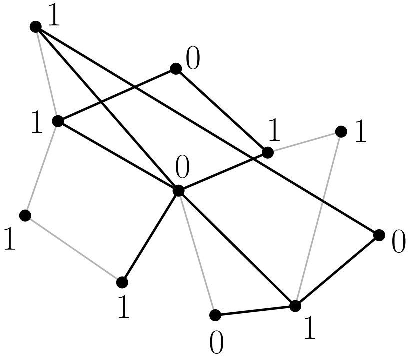

$\frac{1}{2}$-approximation in 0 communication rounds !

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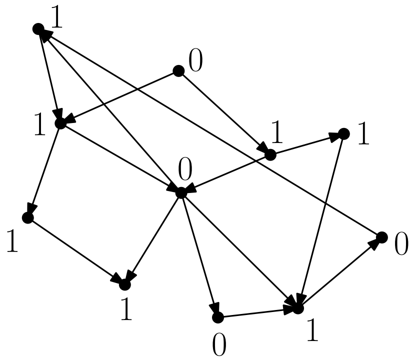

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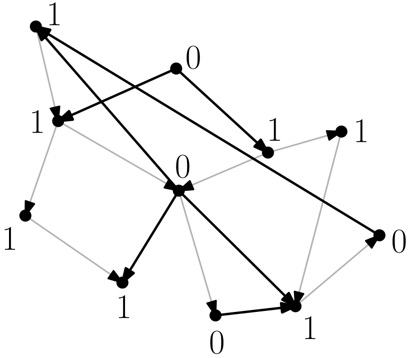

$\frac{1}{4}$-approximation in 0 communication rounds !

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State-of-the-art in the CONGEST model:

Problem	approx.	rounds	rand/det	source
MaxCut	$1/2$	0	rand.	folklore
MaxDiCut	$1/4$	0	rand.	folklore
MaxCut d-regular graphs + triangle-free	$\approx 1/2 + 0.28/\sqrt{d}$	$O(1)$	rand.	[1]
MaxCut	$1/2 - \epsilon$	$O(\log^* n)$	det.	[2]
MaxDiCut	$1/3 - \epsilon$	$O(\log^* n)$	det.	[2]
MaxDiCut	$1/2 - \epsilon$	$O(\epsilon^{-1})$	rand.	[2]

[1] J. Hirvonen, J. Rybicki, S. Schmid, J. Suomela (Electron. J. of Combin. 2017)

[2] K. Kawarabayashi, G. Schwartzman (DISC 18)

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Our negative results:

Theorem

Let $d > 0$ be an integer.

- If d is even, any deterministic algorithm in the LOCAL model that guarantees a constant factor approximation for MaxCut on the class of bipartite d-regular graphs n-vertex graphs runs in $\Omega(\log^* n)$ rounds.
- If d is odd, then for any $\epsilon > 0$, any $\left(\frac{1}{d} + \epsilon\right)$-approximation requires $\Omega(\log^* n)$ rounds in the same setting.

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For the oriented cut problem (MaxDiCut), the results are the same except that $\frac{1}{d}$ is replaced by $\frac{2}{d}$ in the odd case.

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- For the MaxDiCut, there exists a deterministic $\frac{2}{d+1/d}$-approximation on d-regular graphs running in 0 round in the CONGEST model.
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These algorithms are very simple and even do not require any communication if the orientation of edges is given (PO model for instance).

Proof of the lower bound

Theorem (Ramsey (1930))

Consider a hypergraph $G = (V, E)$ where edges are sets of r vertices. Fix $m > 0$ then for any mapping $f : E \mapsto \{0, 1\}$ there exists a n big enough such that there is a monochromatic induced subgraph of size m.

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Here a neighborhood is given by the edge set, the set of IDs and the permutation of IDs on vertices

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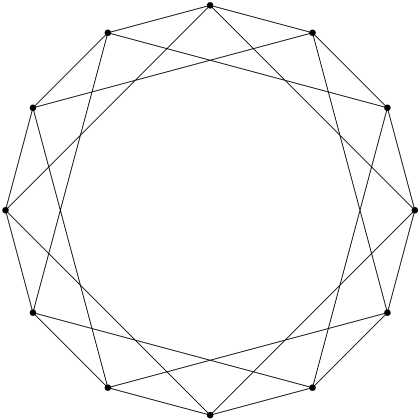
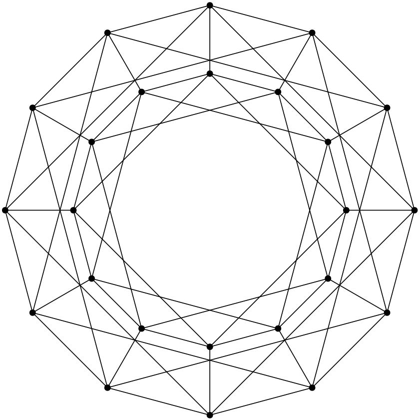


Figure: d even



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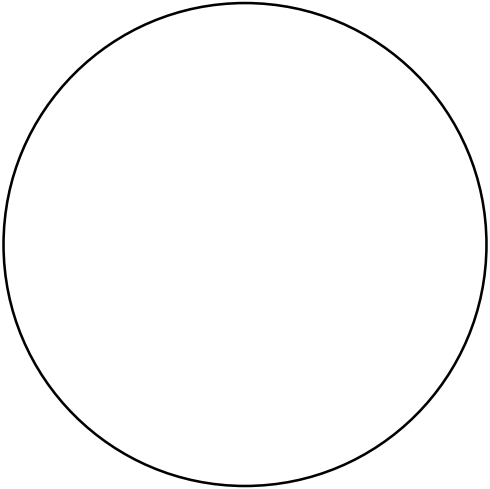

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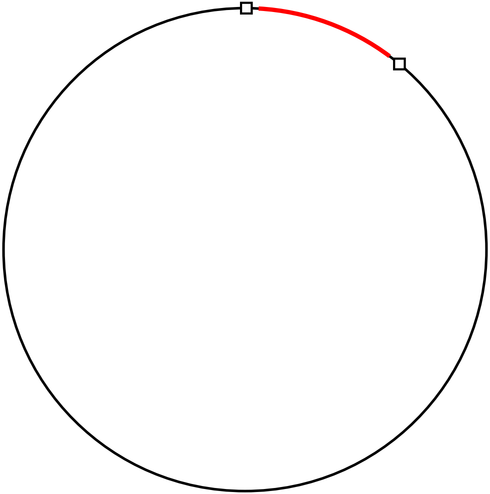

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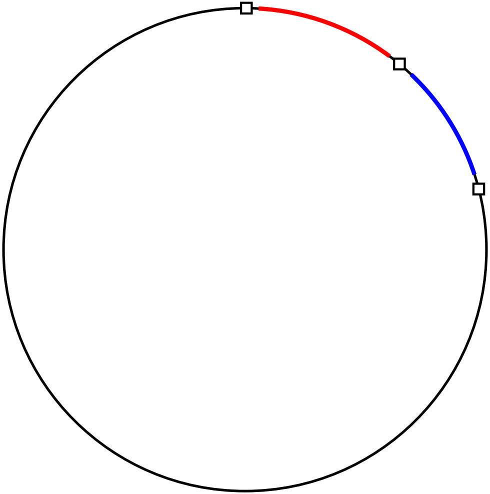

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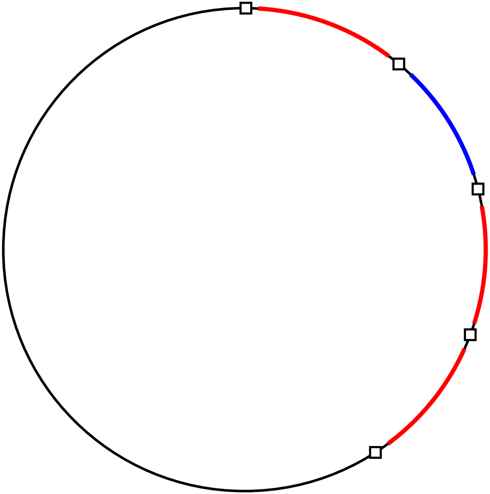

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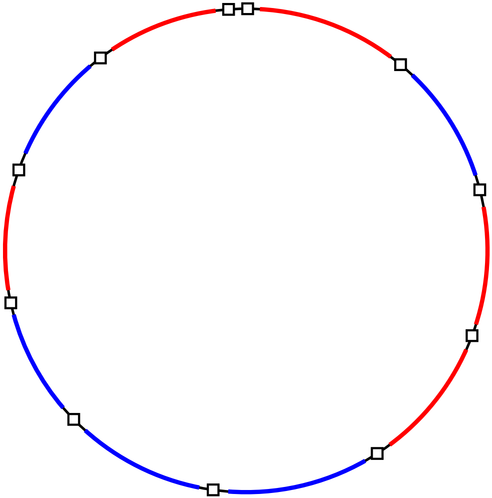

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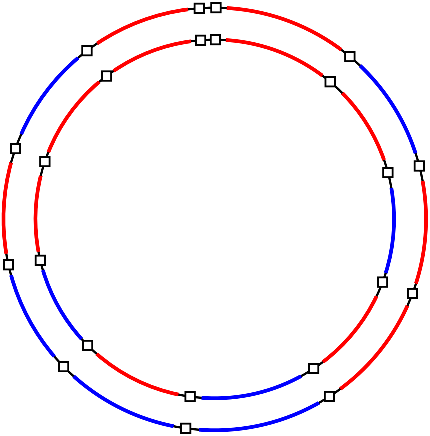

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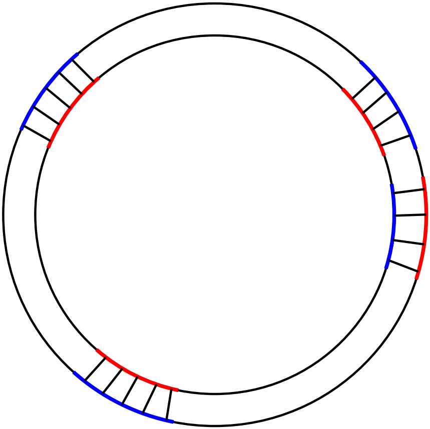

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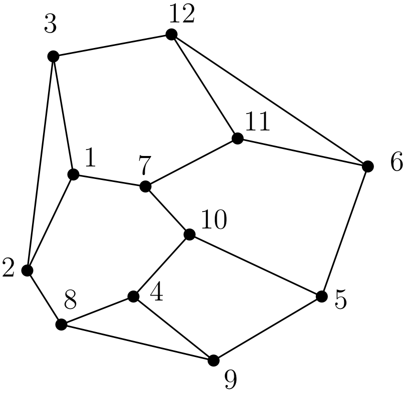
Unoriented case with d odd: every vertex v collects the list $L(v)$ of IDs of its neighbors. Let $m(v)$ be the median value of $L(v)$



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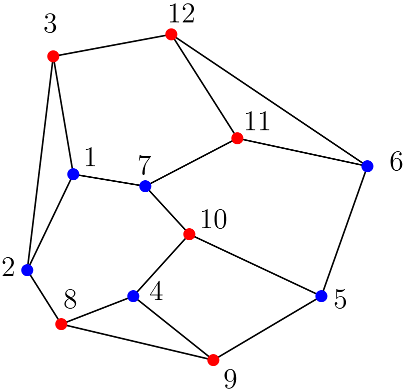

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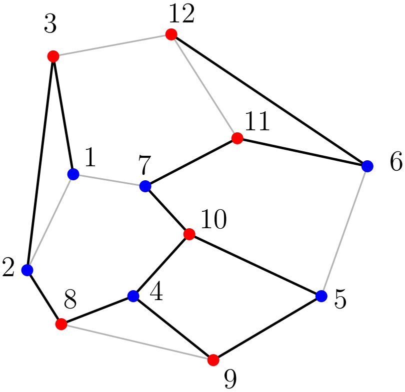

v chooses the left side if and only if $ID(v) > m(v) \implies \frac{1}{d}$-approximation

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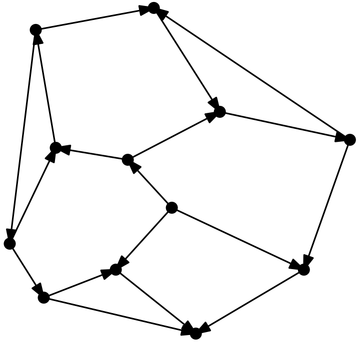
Oriented case with d odd: a vertex chooses the right side if and only if it has more ingoing than outgoing edges



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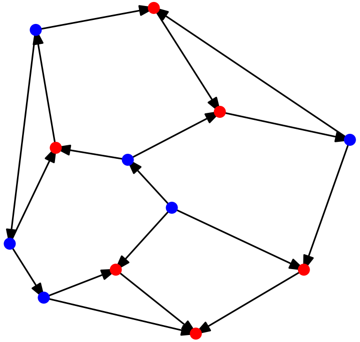

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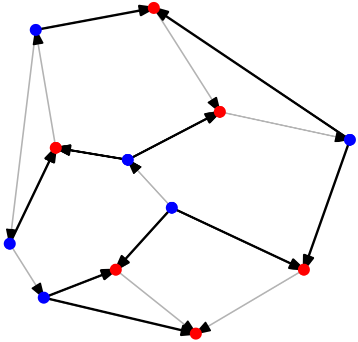

$\Longrightarrow$ $\frac{2}{d+1/d}$-approximation (the ratio is tight).

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In a given cut, a vertex v is said to be unstable if and only if all its neighbors are on the same side of the cut as v

For 2 rounds, every unstable vertex changes side. $\implies \left(\frac{2}{d+1/d} + \alpha\right)$ approximation ratio

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Is the upper bound on approximation ratio $\frac{2}{d}$ matched ?

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Maximal cut behaves very differently: $\Omega(\log n)$ deterministic rounds and $\Omega(\log \log n)$ randomized rounds (A. Balliu, J. Hirvonen, C. Lenzen, D. Olivetti, J. Suomela (SIROCCO 2019))